we are repeatedly picking objects from the same category, we need to be more careful.

Let's consider an example. At the beginning of math class every day, Mr. Smith selects students to write up homework problems on the board. These problems are then discussed as a class. There are 26 students in Mr. Smith's math class, and he randomly selects a student to write up each of the first five problems. How many different ways can students be assigned to the problems?

This problem is more complicated than the sandwich or candy problems because we are selecting objects from the same group each time. Mr. Smith is not selecting students from five different classes or from different groups within his class. In this way, the problem is more like the license plate problem. In that problem, we were selecting repeatedly from two different groups. Here we are repeatedly selecting objects from only one group, the class of 26 students.

The first thing we have to determine is if any student can be selected more than once. In our imaginary list of all possible arrangements, is *Fred, Fred, Fred, Fred, Fred* an allowable selection? Or if Fred is selected to write up the first problem, is he "safe" from writing up problems #2 – 5?

This is an important distinction and can drastically impact the answer to the question. Fortunately, this does not significantly alter the way we think about the problem, just the answer we get. Hopefully whether or not objects can be repeated is clear from the context and description of the problem. Sometimes, the phrases "with replacement" or "without replacement" are used to clarify whether or not an object from a category can be selected more than once. This example will be presented for you to solve in the review problems at the end of this section. Let's now work through a few examples here.

EXAMPLE 1.2a: How many different ways can three cards be drawn with replacement from a standard 52-card deck?

SOLUTION: Since the cards are drawn with replacement, each card is put back into the deck before the next card is drawn. So, the same card could be drawn all three times, and there are 52 choices for the first card, 52 choices for the second card, and 52 choices for the third card. Therefore, there are $52 \times 52 \times 52 =$ **140,608** different ways three cards can be drawn with replacement.

EXAMPLE 1.2b: How many different ways can three cards be drawn without replacement from a standard 52-card deck, assuming the order of drawing cards matters?

SOLUTION: Since the cards are drawn without replacement, each card is kept when the next card is drawn. This means there are 52 choices for the first card, but only 51 choices for the second card, and 50 choices for the third card. Therefore, there are $52 \times 51 \times 50 =$ **132,600** different ways three different cards can be drawn without replacement.

There are two important things to consider from this pair of problems. The first is the difference in the answers. Although there are more ways to draw three cards with replacement than without, this difference is perhaps not as large as expected. Because there are so many cards to choose from, and we are selecting so few of them, most

$+ \binom{10}{7} \times x^7 + \binom{10}{6} \times x^6 + \binom{10}{5} \times x^5 + \binom{10}{4} \times x^4 + \binom{10}{3} \times x^3 + \binom{10}{2} \times x^2 + \binom{10}{1} \times x^1 + \binom{10}{0} \times x^0.$

This is, of course, a bit cumbersome to write, so writing this polynomial in sigma form makes sense.

Therefore, $(x + 1)^{10} = \sum_{k=0}^{10} \binom{10}{k} \times x^k$.

This is a very nice generalization. Let's try another expansion like this, but one that is a little more complicated, to see if the same type of mathematical structure applies.

EXAMPLE 2.3c: What is $(2x + 5)^{15}$?

SOLUTION: Rather than write out all fifteen terms of $2x + 5$, we'll imagine them written out in a long line. As in Example 2.3b, each term in the product is generated by selecting either $2x$ or 5 from each of the fifteen terms and multiplying them together.

So, $(2x) \times 5 \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x)$ is one possible term in the expansion, and this term would be grouped with the other x^{11} terms. Let's go through a few powers of x and try to determine the coefficients.

The highest power of x will be x^{15}, and this will occur when the $2x$ is selected from all fifteen terms. $(2x)^{15} = 2^{15} \times x^{15}$, and so the coefficient of x^{15} will be 2^{15}, or 32,768.

The next highest power of x is x^{14}, and these will be generated by selecting the $2x$ from every term except one, from which a 5 will be selected. Each of these terms will therefore look like:

$(2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times 5$,

and so each will equal $(2x)^{14} \times 5 = 2^{14} \times 5 \times x^{14}$, or $81920 \times x^{14}$. But how many of these terms will be created? With fifteen spots and needing to select 14 of them as $2x$ (or one of them as 5), there should be 15 ways to do this. Note that $\binom{15}{14} = \binom{15}{1} = 15$, because selecting 14 spots for the $2x$ automatically selects the spot for the 5, and vice versa. Therefore, the final coefficient of x^{14} will be $81,920 \times 15 = 1,228,800$.

Sometimes the numerical values disguise rather than assist generalization. Instead of thinking about this as $1228800 \times x^{14}$, let's keep track of where it came from: $\binom{15}{14} \times (2x)^{14} \times 5$. Again, this calculation makes sense: when selecting $2x$ from 14 of the 15 terms, $2x$ will be raised to the 14th power, and 5 will be selected from the last term. This selection can then occur in $\binom{15}{14}$ ways, since there are 15 terms, and we are selecting the $2x$ from 14 of them.

What about x^{13} ? These are generated by selecting the $2x$ from 13 of the 15 terms. For example:

$(2x) \times 5 \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x)$. Each of these terms will therefore equal $(2x)^{13} \times 5^2$. But how many of these terms will there be? With 15 spots, selecting 13 of them to be $2x$ needs to occur without replacement, and the order of selection does not matter. So this can we are repeatedly picking objects from the same category, we need to be more careful.

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